Stochastic Volterra equations driven by cylindrical Wiener process

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Abstract

In this paper, stochastic Volterra equations driven by cylindrical Wiener process in Hilbert space are investigated. Sufficient conditions for existence of strong solutions are given. The key role is played by convergence of α-times resolvent families.

1 Introduction

Let H be a separable Hilbert space with a norm $|\cdot|_H$ and A be a closed linear unbounded operator with dense domain $D(A) \subset H$ equipped with the graph norm $|\cdot|_{D(A)}$. The purpose of this paper is to study the existence of strong solutions for a class of stochastic Volterra equations of the form

$$X(t) = X_0 + \int_0^t a(t-\tau)AX(\tau)d\tau + \int_0^t \Psi(\tau)\, dW(\tau), \quad t \geq 0, \tag{1}$$

where $a(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$, $\alpha > 0$, and W, Ψ are appropriate stochastic processes. It is well known that there are several situations that can be modeled by stochastic Volterra equations (see e.g. [7, Section 3.4] and references therein). We note that stochastic Volterra equations driven by white noise have been studied in [3] among other authors. A similar equation and very related to our case appears first studied in [2]. Here we are interested in the study of strong solutions when equation (1) is driven by a cylindrical Wiener process W.

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When $a(t)$ is a completely positive function, sufficient conditions for existence of strong solutions for (1) were obtained in [9]. This was done using a method which involves the use of a resolvent family associated to the deterministic version of equation (1):

$$u(t) = \int_0^t a(t-\tau)\, Au(\tau)d\tau + f(t), \quad t \geq 0, \tag{2}$$

where f is an H-valued function.

However, there are two kinds of problems that arise when we study (1). On the one hand, the kernels $\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ are α-regular and $\frac{\alpha\pi}{2}$ -sectorial but not completely positive functions for $\alpha > 1$, so e.g. the results in [9] cannot be used directly for $\alpha > 1$. On the other hand, for $\alpha \in (0,1)$, we have a singularity of the kernel in $t = 0$. This fact strongly suggests the use of α-times resolvent families associated to equation (2). This new tool appears carefully studied in [1] as well as their relationship with fractional derivatives. For convenience of the reader, we provide below the main results on α-times resolvent families to be used in this paper.

Our second main ingredient to obtain strong solutions of (1) relies on approximation of α-times resolvent families. This kind of result was very recently formulated by Li and Zheng [10]. It enables us to prove a key result on convergence of α-times resolvent families (see Theorem 2 below). Then we can follow the methods employed in [9] to obtain existence of strong solution for the stochastic equation (1) (see Theorem 4).

Our plan for the paper is the following. In section 2 we formulate the deterministic results which will play the key role for the paper. Section 3 is devoted to weak and mild solutions while in section 4 we provide strong solution to (1). More precisely, we give sufficient condition for a stochastic convolution to be a strong solution to (1).

2 Convergence of α-times resolvent families

In this section we formulate the main deterministic results on convergence of resolvents. We denote

$$g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha > 0, \quad t > 0,$$

where Γ is the gamma function.

By $S_\alpha(t),\ t \geq 0$, we denote the family of α-times resolvent families corresponding to the Volterra equation (2), if it exists, and defined as follows.

Definition 1 *(see [1])*
A family $(S_\alpha(t))_{t\geq 0}$ of bounded linear operators in a Banach space B is called α-`times resolvent family` *for (2) if the following conditions are satisfied:*

1. *$S_\alpha(t)$ is strongly continuous on $\mathbb{R}_+$ and $S_\alpha(0) = I$;*

2. *$S_\alpha(t)$ commutes with the operator A, that is, $S_\alpha(t)(D(A)) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;*

3. *the following* `resolvent equation` *holds*

$$S_\alpha(t)x = x + \int_0^t g_\alpha(t-\tau)AS_\alpha(\tau)xd\tau \tag{3}$$

for all $x \in D(A),\ t \geq 0$.

Necessary and sufficient conditions for existence of the α-times resolvent family have been studied in [1]. Observe that the α-times resolvent family corresponds to a C_0-semigroup in case $\alpha = 1$ and a cosine family in case $\alpha = 2$. In consequence, when $1 < \alpha < 2$ such resolvent families interpolate C_0-semigroups and cosine functions. In particular, for $A = \Delta$, the integrodifferential equation corresponding to such resolvent family interpolates the heat equation and the wave equation (see [6]).

Definition 2 *An α-times resolvent family $(S_\alpha(t))_{t\geq 0}$ is called* `exponentially bounded` *if there are constants $M \geq 1$ and $\omega \geq 0$ such that*

$$\|S_\alpha(t)\| \leq Me^{\omega t}, \quad t \geq 0. \tag{4}$$

If there is the α-times resolvent family $(S_\alpha(t))_{t\geq 0}$ for A and satisfying (4), we write $A \in \mathcal{C}^\alpha(M,\omega)$. Also, set $\mathcal{C}^\alpha(\omega) := \cup_{M\geq 1}\mathcal{C}^\alpha(M,\omega)$ and $\mathcal{C}^\alpha := \cup_{\omega\geq 0}\mathcal{C}^\alpha(\omega)$.

Remark 1 It was proved by Bazhlekova [1, Theorem 2.6] that if $A \in \mathcal{C}^\alpha$ for some $\alpha > 2$, then A is bounded.

The following subordination principle is very important in the theory of α-times resolvent families (see [1, Theorem 3.1]).

Theorem 1 *Let $0 < \alpha < \beta \leq 2, \gamma = \alpha/\beta, \omega \geq 0$. If $A \in \mathcal{C}^\beta(\omega)$ then $A \in \mathcal{C}^\alpha(\omega^{1/\gamma})$ and the following representation holds*

$$S_\alpha(t)x = \int_0^\infty \varphi_{t,\gamma}(s)S_\beta(s)xds, \quad t > 0, \tag{5}$$

where $\varphi_{t,\gamma}(s) := t^{-\gamma}\Phi_\gamma(st^{-\gamma})$ and $\Phi_\gamma(z)$ is the Wright function defined as

$$\Phi_\gamma(z) := \sum_{n=0}^\infty \frac{(-z)^n}{n!\,\Gamma(-\gamma n + 1 - \gamma)}, \quad 0 < \gamma < 1. \tag{6}$$

Remark 2 (i) We recall that the Laplace transform of the Wright function corresponds to $E_\gamma(-z)$ where E_γ denotes the Mittag-Leffler function. In particular, $\Phi_\gamma(z)$ is a probability density function.

(ii) Also we recall from [1, (2.9)] that the continuity in $t \geq 0$ of the Mittag-Leffler function together with the asymptotic behavior of it, imply that for $\omega \geq 0$ there exists a constant $C > 0$ such that

$$E_\alpha(\omega t^\alpha) \leq C e^{\omega^{1/\alpha} t}, \quad t \geq 0, \ \alpha \in (0,2). \tag{7}$$

As we have already written, in this paper the results concerning convergence of α-times resolvent families in a Banach space B will play the key role. Using a very recent result due to Li and Zheng [10] we are able to prove the following theorem.

Theorem 2 *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ in a Banach space B such that*

$$\|T(t)\| \leq M e^{\omega t}, \quad t \geq 0. \tag{8}$$

Then, for each $0 < \alpha < 1$ we have $A \in \mathcal{C}^\alpha(M, \omega^{1/\alpha})$. Moreover, there exist bounded operators A_n and α-times resolvent families $S_{\alpha,n}(t)$ for A_n satisfying $||S_{\alpha,n}(t)|| \leq MCe^{(2\omega)^{1/\alpha}t}$, for all $t \geq 0, \ n \in \mathbb{N}$, and

$$S_{\alpha,n}(t)x \to S_\alpha(t)x \quad \text{as} \quad n \to +\infty \tag{9}$$

for all $x \in B, \ t \geq 0$. Moreover, the convergence is uniform in t on every compact subset of $\mathbb{R}_+$.

Proof Since A is the generator of a C_0 semigroup satisfying (8), we have $A \in \mathcal{C}^1(\omega)$. Hence, the first assertion follows directly from Theorem 1, that is, for each $0 < \alpha < 1$ there is an α-times resolvent family $(S_\alpha(t))_{t\geq 0}$ for A given by

$$S_\alpha(t)x = \int_0^\infty \varphi_{t,\alpha}(s) T(s) x ds, \quad t > 0. \tag{10}$$

Since A generates a C_0-semigroup, the resolvent set $\rho(A)$ of A contains the ray $[w, \infty)$ and

$$||R(\lambda, A)^k|| \leq \frac{M}{(\lambda - w)^k} \qquad \text{for } \lambda > w, \qquad k \in \mathbb{N}.$$

Define

$$A_n := nAR(n,A) = n^2 R(n,A) - nI, \qquad n > w, \tag{11}$$

the *Yosida approximation* of A.

Then

$$\begin{aligned} ||e^{tA_n}|| &= e^{-nt} ||e^{n^2 R(n,A)t}|| \leq e^{-nt} \sum_{k=0}^\infty \frac{n^{2k} t^k}{k!} ||R(n,A)^k|| \\ &\leq Me^{(-n + \frac{n^2}{n-w})t} = Me^{\frac{nwt}{n-w}}. \end{aligned}$$

Hence, for $n > 2w$ we obtain

$$||e^{A_n t}|| \le Me^{2wt}. \tag{12}$$

Next, since each A_n is bounded, it follows also from Theorem 1 that for each $0 < \alpha < 1$ there exists an α-times resolvent family $(S_{\alpha,n}(t))_{t\ge 0}$ for A_n given as

$$S_{\alpha,n}(t) = \int_0^\infty \varphi_{t,\alpha}(s) e^{sA_n} ds, \quad t > 0. \tag{13}$$

By (12) and Remark 2(i) it follows that

$$\begin{aligned} \|S_{\alpha,n}(t)\| &\le \int_0^\infty \varphi_{t,\alpha}(s)\|e^{sA_n}\|ds \\ &\le M\int_0^\infty \varphi_{t,\alpha}(s)e^{2\omega s}ds = M\int_0^\infty \Phi_\alpha(\tau)e^{2\omega t^\alpha \tau}d\tau = ME_\alpha(2\omega t^\alpha), \quad t \ge 0. \end{aligned}$$

This together with Remark 2(ii), gives

$$\|S_{\alpha,n}(t)\| \le MCe^{(2\omega)^{1/\alpha}t}, \quad t \ge 0. \tag{14}$$

Now, we recall the fact that $R(\lambda, A_n)x \to R(\lambda, A)x$ as $n \to \infty$ for all λ sufficiently large (see e.g. [11, Lemma 7.3]), so we can conclude from [10, Theorem 4.2] that

$$S_{\alpha,n}(t)x \to S_\alpha(t)x \quad \text{as} \quad n \to +\infty \tag{15}$$

for all $x \in B$, uniformly for t on every compact subset of $\mathbb{R}_+$. ■

An analogous result can be proved in the case when A is the generator of a strongly continuous cosine family.

Theorem 3 *Let A be the generator of a C_0-cosine family $(T(t))_{t\ge 0}$ in a Banach space B. Then, for each $0 < \alpha < 2$ we have $A \in \mathcal{C}^\alpha(M, \omega^{2/\alpha})$. Moreover, there exist bounded operators A_n and α-times resolvent families $S_{\alpha,n}(t)$ for A_n satisfying $||S_{\alpha,n}(t)|| \le MCe^{(2\omega)^{1/\alpha}t}$, for all $t \ge 0,\ n \in \mathbb{N}$, and*

$$S_{\alpha,n}(t)x \to S_\alpha(t)x \quad \textit{as} \quad n \to +\infty$$

for all $x \in B,\ t \ge 0$. Moreover, the convergence is uniform in t on every compact subset of $\mathbb{R}_+$.

In the following, we denote by $\Sigma_\theta(\omega)$ the open sector with vertex $\omega \in \mathbb{R}$ and opening angle 2θ in the complex plane which is symmetric with respect to the real positive axis, i.e.

$$\Sigma_\theta(\omega) := \{\lambda \in \mathbb{C} : |arg(\lambda - \omega)| < \theta\}.$$

We recall from [1, Definition 2.13] that an α-times resolvent family $S_\alpha(t)$ is called `analytic` if $S_\alpha(t)$ admits an analytic extension to a sector Σ_{θ_0} for some $\theta_0 \in (0, \pi/2]$. An α-times analytic resolvent family is said to be of `analyticity type` (θ_0, ω_0) if for each $\theta < \theta_0$ and $\omega > \omega_0$ there is $M = M(\theta, \omega)$ such that

$$\|S_\alpha(t)\| \le M e^{\omega Ret}, \quad t \in \Sigma_\theta.$$

The set of all operators $A \in \mathcal{C}^\alpha$ generating α-times analytic resolvent families $S_\alpha(t)$ of type (θ_0, ω_0) is denoted by $\mathcal{A}^\alpha(\theta_0, \omega_0)$. In addition, denote $\mathcal{A}^\alpha(\theta_0) := \bigcup\{\mathcal{A}^\alpha(\theta_0, \omega_0); \omega_0 \in \mathbb{R}_+\}$, $\quad \mathcal{A}^\alpha := \bigcup\{\mathcal{A}^\alpha(\theta_0); \theta_0 \in (0, \pi/2]\}$. For $\alpha = 1$ we obtain the set of all generators of analytic semigroups.

Remark 3 We note that the spatial regularity condition $\mathcal{R}(S_\alpha(t)) \subset D(A)$ for all $t > 0$ is satisfied by α-times resolvent families whose generator A belongs to the set $\mathcal{A}^\alpha(\theta_0, \omega_0)$ where $0 < \alpha < 2$ (see [1, Proposition 2.15]). In particular, setting $\omega_0 = 0$ we have that $A \in \mathcal{A}^\alpha(\theta_0, 0)$ if and only if $-A$ is a positive operator with spectral angle less or equal to $\pi - \alpha(\pi/2 + \theta)$. Note that such condition is also equivalent to the following

$$\Sigma_{\alpha(\pi/2+\theta)} \subset \rho(A) \text{ and } \|\lambda(\lambda I - A)^{-1}\| \le M, \quad \lambda \in \Sigma_{\alpha(\pi/2+\theta)}. \tag{16}$$

The above considerations give us the following remarkable corollary.

Corollary 1 *Suppose A generates an analytic semigroup of angle $\pi/2$ and $\alpha \in (0, 1)$. Then A generates an α-times analytic resolvent family.*

Proof Since A generates an analytic semigroup of angle $\pi/2$ we have

$$\|\lambda(\lambda I - A)^{-1}\| \le M, \quad \lambda \in \Sigma_{\pi-\epsilon}.$$

Then the condition (16) (see also [1, Corollary 2.16]) implies $A \in \mathcal{A}^\alpha(\min\{\frac{2-\alpha}{2\alpha}\pi, \frac{1}{2}\pi\}, 0)$, $\alpha \in (0, 2)$, that is A generates an α-times analytic resolvent family. ■

In the sequel we will use the following assumptions concerning Volterra equations:

(A1) A is the generator of C_0-semigroup and $\alpha \in (0, 1)$; or

(A2) A is the generator of a strongly continuous cosine family and $\alpha \in (0, 2)$.

Observe that (A2) implies (A1) but not vice versa.

3 Weak vs. mild solutions

Assume that H and U are separable Hilbert spaces. Let the cylindrical Wiener process W be defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, P)$, with the positive symmetric covariance operator $Q \in L(U)$. This is known that the process W takes values in some superspace of U. (For more details concerning cylindrical Wiener process we refer to [4] or [8].)

We define the subspace $U_0 := Q^{1/2}(U)$ of the space U, endowed with the inner product $\langle u,v\rangle_{U_0} := \langle Q^{-1/2}u, Q^{-1/2}v\rangle_U$. The set $L_2^0 := L_2(U_0, H)$ of all Hilbert-Schmidt operators from U_0 into H, equipped with the norm $|C|_{L_2(U_0,H)} := (\sum_{k=1}^{+\infty} |Cf_k|_H^2)^{1/2}$, where $\{f_k\} \subset U_0$ is an orthonormal basis of U_0, is a separable Hilbert space. We assume that Ψ belongs to the class of measurable L_2^0-valued processes.

By $\mathcal{N}^2(0,T;L_2^0)$ we denote a Hilbert space of all L_2^0-predictable processes Ψ such that $||\Psi||_T < +\infty$, where

$$
\begin{aligned}
||\Psi||_T &:= \left\{ \mathbb{E}\left(\int_0^T |\Psi(\tau)|_{L_2^0}^2 \, d\tau \right) \right\}^{\frac{1}{2}} \\
&= \left\{ \mathbb{E} \int_0^T \left[\mathrm{Tr}(\Psi(\tau)Q^{\frac{1}{2}})(\Psi(\tau)Q^{\frac{1}{2}})^* \right] d\tau \right\}^{\frac{1}{2}}.
\end{aligned}
$$

We shall use the following PROBABILITY ASSUMPTIONS (abbr. (PA)):

1. X_0 is an H-valued, $\mathcal{F}_0$-measurable random variable;
2. $\Psi \in \mathcal{N}^2(0,T;L_2^0)$ and the interval $[0,T]$ is fixed.

Definition 3 *Assume that (PA) hold. An H-valued predictable process $X(t)$, $t \in [0,T]$, is said to be a* `strong solution` *to (1), if X takes values in $D(A)$, P-a.s.,*

$$
\text{for any } t \in [0,T], \quad \int_0^t |g_\alpha(t-\tau)AX(\tau)|_H \, d\tau < +\infty, \quad P-a.s., \quad \alpha > 0, \tag{17}
$$

and for any $t \in [0,T]$ the equation (1) holds P-a.s.

Let A^* denote the adjoint of A with a dense domain $D(A^*) \subset H$ and the graph norm $|\cdot|_{D(A^*)}$.

Definition 4 *Let (PA) hold. An H-valued predictable process $X(t)$, $t \in [0,T]$, is said to be a* `weak solution` *to (1), if $P(\int_0^t |g_\alpha(t-\tau)X(\tau)|_H d\tau < +\infty) = 1$, $\alpha > 0$, and if for all $\xi \in D(A^*)$ and all $t \in [0,T]$ the following equation holds*

$$
\langle X(t), \xi\rangle_H = \langle X_0, \xi\rangle_H + \langle \int_0^t g_\alpha(t-\tau)X(\tau)\, d\tau, A^*\xi\rangle_H + \langle \int_0^t \Psi(\tau)dW(\tau), \xi\rangle_H, \quad P\text{–a.s.}
$$

Definition 5 *Assume that X_0 is $\mathcal{F}_0$-measurable random variable. An H-valued predictable process $X(t),\ t\in[0,T]$, is said to be a* `mild solution` *to the stochastic Volterra equation (1), if* $\mathbb{E}(\int_0^t |S_\alpha(t-\tau)\Psi(\tau)|^2_{L_2^0}\,d\tau) < +\infty,\ \alpha>0$, *for $t\le T$ and, for arbitrary $t\in[0,T]$,*

$$X(t) = S_\alpha(t)X_0 + \int_0^t S_\alpha(t-\tau)\Psi(\tau)\,dW(\tau), \quad P-a.s. \tag{18}$$

where $S_\alpha(t)$ is the α-times resolvent family.

We will use the following result.

Proposition 1 *(see, e.g. [4, Proposition 4.15])*
Assume that A is a closed linear unbounded operator with the dense domain $D(A)\subset H$. Let $\Phi(t), t\in[0,T]$, be an $L_2(U_0,H)$-predictable process. If $\Phi(t)\in D(A),\quad P-a.s.$ for all $t\in[0,T]$ and

$$P\left(\int_0^T |\Phi(s)|^2_{L_2^0}\,ds<\infty\right)=1,\quad P\left(\int_0^T |A\Phi(s)|^2_{L_2^0}\,ds<\infty\right)=1,$$

then $\quad P\left(\int_0^T \Phi(s)\,dW(s)\in D(A)\right)=1$ *and*

$$A\int_0^T \Phi(s)\,dW(s) = \int_0^T A\Phi(s)\,dW(s), \quad P-a.s.$$

We define the stochastic convolution

$$W_\alpha^\Psi(t) := \int_0^t S_\alpha(t-\tau)\Psi(\tau)\,dW(\tau), \tag{19}$$

where $\Psi\in\mathcal{N}^2(0,T;L_2^0)$. Because α-times resolvent families $S_\alpha(t),\ t\ge 0$, are bounded, then $S_\alpha(t-\cdot)\Psi(\cdot)\in\mathcal{N}^2(0,T;L_2^0)$, too.

Analogously like in [8], we can formulate the following result.

Proposition 2 *Assume that $S_\alpha(t), t\ge 0$, are the resolvent operators to (2). Then, for any process $\Psi\in\mathcal{N}^2(0,T;L_2^0)$, the convolution $W_\alpha^\Psi(t),\ t\ge 0,\ \alpha>0$, given by (19) has a predictable version. Additionally, the process $W_\alpha^\Psi(t),\ t\ge 0,\ \alpha>0$, has square integrable trajectories.*

Under some conditions a mild solution to Volterra equations is a weak solution and vice versa, see [8, Propositions 4 and 5].

Now, we can prove that a mild solution to the equation (1) is a weak solution to (1).

Proposition 3 *If $\Psi \in \mathcal{N}^2(0,T;L_2^0)$ and $\Psi(\cdot,\cdot)(U_0) \subset D(A)$, P-a.s., then the stochastic convolution $W_\alpha^\Psi(t)$, $t \geq 0$, $\alpha > 0$, given by (19), fulfills the equation*

$$\langle W_\alpha^\Psi(t), \xi\rangle_H = \int_0^t \langle g_\alpha(t-\tau) W_\alpha^\Psi(\tau), A^*\xi\rangle_H + \int_0^t \langle \xi, \Psi(\tau) dW(\tau)\rangle_H, \quad \alpha \in (0,2), \tag{20}$$

for any $t \in [0,T]$ and $\xi \in D(A^*)$.

Proof Let us notice that the process W_α^Ψ has integrable trajectories. For any $\xi \in D(A^*)$ we have

$$\begin{aligned}
\int_0^t \langle g_\alpha(t-\tau) W_\alpha^\Psi(\tau), A^*\xi\rangle_H d\tau \quad &\equiv \quad (\text{from } (19)) \\
&\equiv \int_0^t \langle g_\alpha(t-\tau) \int_0^\tau S_\alpha(\tau-\sigma)\Psi(\sigma) dW(\sigma), A^*\xi\rangle_H d\tau = \\
(\text{from Dirichlet's formula and } & \text{ stochastic Fubini's theorem)} \\
&= \int_0^t \langle \left[\int_\sigma^t g_\alpha(t-\tau) S_\alpha(\tau-\sigma) d\tau\right] \Psi(\sigma) dW(\sigma), A^*\xi\rangle_H \\
&= \langle \int_0^t \left[\int_0^{t-\sigma} g_\alpha(t-\sigma-z) S_\alpha(z) dz\right] \Psi(\sigma) dW(\sigma), A^*\xi\rangle_H \\
(\text{where } z := \tau - \sigma \quad & \text{ and from definition of convolution)} \\
&= \langle \int_0^t A[(g_\alpha \star S_\alpha)(t-\sigma)] \Psi(\sigma) dW(\sigma), \xi\rangle_H = \\
(\text{from the resolvent equation (3)} \quad & \text{ because } A(g_\alpha \star S_\alpha)(t-\sigma)x = (S_\alpha(t-\sigma) - I)x, \\
& \text{ where } x \in D(A)) \\
&= \langle \int_0^t [S_\alpha(t-\sigma) - I] \Psi(\sigma) dW(\sigma), \xi\rangle_H = \\
= \langle \int_0^t S_\alpha(t-\sigma) \Psi(\sigma) dW(\sigma), \xi\rangle_H \quad &- \quad \langle \int_0^t \Psi(\sigma) dW(\sigma), \xi\rangle_H .
\end{aligned}$$

Hence, we obtained the following equation

$$\langle W_\alpha^\Psi(t), \xi\rangle_H = \int_0^t \langle g_\alpha(t-\tau) W_\alpha^\Psi(\tau), A^*\xi\rangle_H d\tau + \int_0^t \langle \xi, \Psi(\tau) dW(\tau)\rangle_H$$

for any $\xi \in D(A^*)$. ■

Immediately from the equation (20) we deduce the following result.

Corollary 2 *If A is a bounded operator and $\Psi \in \mathcal{N}^2(0,T;L_2^0)$, then the following equality holds*

$$W_\alpha^\Psi(t) = \int_0^t g_\alpha(t-\tau) A W_\alpha^\Psi(\tau) d\tau + \int_0^t \Psi(\tau) dW(\tau), \tag{21}$$

for $t \in [0,T]$, $\alpha > 0$.

Remark 4 The formula (21) says that the convolution $W_\alpha^\Psi(t)$, $t \geq 0$, $\alpha > 0$, is a strong solution to (1) with $X_0 \equiv 0$ if the operator A is bounded.

4 Strong solutions

In this section we provide sufficient conditions under which the stochastic convolution $W_\alpha^\Psi(t)$, $t \geq 0$, $\alpha > 0$, defined by (19) is a strong solution to the equation (1).

Lemma 1 *Let A be a closed linear unbounded operator with dense domain $D(A)$ equipped with the graph norm $|\cdot|_{D(A)}$. Assume that* **(A1)** *or* **(A2)** *holds. If Ψ and $A\Psi$ belong to $\mathcal{N}^2(0,T;L_2^0)$ and in addition $\Psi(\cdot,\cdot)(U_0) \subset D(A)$, P-a.s., then (21) holds.*

Proof Because formula (21) holds for any bounded operator, then it holds for the Yosida approximation A_n of the operator A, too, that is

$$W_{\alpha,n}^\Psi(t) = \int_0^t g_\alpha(t-\tau) A_n W_{\alpha,n}^\Psi(\tau) d\tau + \int_0^t \Psi(\tau) dW(\tau),$$

where

$$W_{\alpha,n}^\Psi(t) := \int_0^t S_{\alpha,n}(t-\tau)\Psi(\tau) dW(\tau).$$

By Proposition 1, we have

$$A_n W_{\alpha,n}^\Psi(t) = A_n \int_0^t S_{\alpha,n}(t-\tau)\Psi(\tau) dW(\tau).$$

By assumption $\Psi \in \mathcal{N}^2(0,T;L_2^0)$. Because the operators $S_{\alpha,n}(t)$ are deterministic and bounded for any $t \in [0,T]$, $\alpha > 0$, $n \in \mathbb{N}$, then the operators $S_{\alpha,n}(t-\cdot)\Psi(\cdot)$ belong to $\mathcal{N}^2(0,T;L_2^0)$, too. In consequence, the difference

$$\Phi_{\alpha,n}(t-\cdot) := S_{\alpha,n}(t-\cdot)\Psi(\cdot) - S_\alpha(t-\cdot)\Psi(\cdot) \tag{22}$$

belongs to $\mathcal{N}^2(0,T;L_2^0)$ for any $t \in [0,T]$, $\alpha > 0$ and $n \in \mathbb{N}$. This means that

$$\mathbb{E}\left(\int_0^t |\Phi_{\alpha,n}(t-\tau)|_{L_2^0}^2 d\tau\right) < +\infty \tag{23}$$

for any $t \in [0,T]$.

Let us recall that the cylindrical Wiener process $W(t)$, $t \geq 0$, can be written in the form

$$W(t) = \sum_{j=1}^{+\infty} f_j\, \beta_j(t), \tag{24}$$

where $\{f_j\}$ is an orthonormal basis of U_0 and $\beta_j(t)$ are independent real Wiener processes. From (24) we have

$$\int_0^t \Phi_{\alpha,n}(t-\tau)\, dW(\tau) = \sum_{j=1}^{+\infty} \int_0^t \Phi_{\alpha,n}(t-\tau)\, f_j\, d\beta_j(\tau). \tag{25}$$

In consequence, from (23)

$$\mathbb{E}\left[\int_0^t \left(\sum_{j=1}^{+\infty} |\Phi_{\alpha,n}(t-\tau)\, f_j|_H^2\right) d\tau\right] < +\infty \tag{26}$$

for any $t \in [0,T]$. Next, from (25), properties of stochastic integral and (26) we obtain for any $t \in [0,T]$,

$$\begin{aligned}
\mathbb{E}\left|\int_0^t \Phi_{\alpha,n}(t-\tau)\, dW(\tau)\right|_H^2 &= \mathbb{E}\left|\sum_{j=1}^{+\infty}\int_0^t \Phi_{\alpha,n}(t-\tau)\, f_j\, d\beta_j(\tau)\right|_H^2 \leq \\
\mathbb{E}\left[\sum_{j=1}^{+\infty}\int_0^t |\Phi_{\alpha,n}(t-\tau)\, f_j|_H^2 d\tau\right] &\leq \mathbb{E}\left[\sum_{j=1}^{+\infty}\int_0^T |\Phi_{\alpha,n}(T-\tau)\, f_j|_H^2 d\tau\right] < +\infty.
\end{aligned}$$

By Theorem 2, the convergence (9) of α-times resolvent families is uniform in t on every compact subset of $\mathbb{R}_+$, particularly on the interval $[0,T]$. Now, we use (9) in the Hilbert space H, so (9) holds for every $x \in H$. Then, for any fixed α and j,

$$\int_0^T |[S_{\alpha,n}(T-\tau) - S_\alpha(T-\tau)]\, \Psi(\tau)\, f_j|_H^2 d\tau \tag{27}$$

tends to zero for $n \to +\infty$. So, summing up our considerations, particularly using (26) and (27) we can write

$$\begin{aligned}
\sup_{t\in[0,T]} \mathbb{E}\left|\int_0^t \Phi_{\alpha,n}(t-\tau) dW(\tau)\right|_H^2 \equiv \sup_{t\in[0,T]} \mathbb{E}\left|\int_0^t [S_{\alpha,n}(t-\tau) - S_\alpha(t-\tau)]\Psi(\tau) dW(\tau)\right|_H^2 \leq \\
\leq \mathbb{E}\left[\sum_{j=1}^{+\infty}\int_0^T |[S_{\alpha,n}(T-\tau) - S_\alpha(T-\tau)]\Psi(\tau)\, f_j|_H^2 d\tau\right] \to 0
\end{aligned}$$

as $n \to +\infty$ for any fixed $\alpha > 0$.

Hence, by the Lebesgue dominated convergence theorem we obtained

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} \mathbb{E}\left|W_{\alpha,n}^\Psi(t) - W_\alpha^\Psi(t)\right|_H^2 = 0. \tag{28}$$

By Proposition 1, $P(W_\alpha^\Psi(t) \in D(A)) = 1$.

For any $n \in \mathbb{N}$, $\alpha > 0$, $t \geq 0$, we have

$$|A_n W_{\alpha,n}^\Psi(t) - A W_\alpha^\Psi(t)|_H \leq N_{n,1}(t) + N_{n,2}(t),$$

where

$$\begin{aligned}
N_{n,1}(t) &:= |A_n W_{\alpha,n}^\Psi(t) - A_n W_\alpha^\Psi(t)|_H, \\
N_{n,2}(t) &:= |A_n W_\alpha^\Psi(t) - A W_\alpha^\Psi(t)|_H = |(A_n - A) W_\alpha^\Psi(t)|_H\,.
\end{aligned}$$

Then

$$|A_n W^{\Psi}_{\alpha,n}(t) - AW^{\Psi}_{\alpha}(t)|^2_H \le N^2_{n,1}(t) + 2N_{n,1}(t)N_{n,2}(t) + N^2_{n,2}(t). \tag{29}$$

Let us study the term $N_{n,1}(t)$. Note that, either in cases **(A1)** or **(A2)** the unbounded operator A generates a semigroup. Then we have for the Yosida approximation the following properties:

$$A_n x = J_n A x \quad \text{for any } x \in D(A), \quad \sup_n ||J_n|| < \infty \tag{30}$$

where $A_n x = nAR(n,A)x = AJ_n x$ for any $x \in H$, with $J_n := nR(n,A)$. Moreover (see [5, Chapter II, Lemma 3.4]):

$$\begin{aligned} \lim_{n\to\infty} J_n x &= x \qquad \text{for any } x \in H, \\ \lim_{n\to\infty} A_n x &= Ax \qquad \text{for any } x \in D(A). \end{aligned} \tag{31}$$

Note that $AS_{\alpha,n}(t)x = S_{\alpha,n}(t)Ax$ for all $x \in D(A)$, since e^{tA_n} commutes with A and A is closed (see (13)). So, by Proposition 1 and again the closedness of A we can write

$$\begin{aligned} A_n W^{\Psi}_{\alpha,n}(t) &\equiv A_n \int_0^t S_{\alpha,n}(t-\tau)\Psi(\tau)dW(\tau) \\ &= J_n \int_0^t AS_{\alpha,n}(t-\tau)\Psi(\tau)dW(\tau) = J_n \left[\int_0^t S_{\alpha,n}(t-\tau)A\Psi(\tau)dW(\tau)\right]. \end{aligned}$$

Analogously,

$$A_n W^{\Psi}_{\alpha}(t) = J_n \left[\int_0^t S_{\alpha}(t-\tau)A\Psi(\tau)dW(\tau)\right].$$

By (30) we have

$$\begin{aligned} N_{n,1}(t) &= |J_n \int_0^t [S_{\alpha,n}(t-\tau) - S_{\alpha}(t-\tau)]A\Psi(\tau)dW(\tau)|_H \\ &\le |\int_0^t [S_{\alpha,n}(t-\tau) - S_{\alpha}(t-\tau)]A\Psi(\tau)dW(\tau)|_H \,. \end{aligned}$$

From assumptions, $A\Psi \in \mathcal{N}^2(0,T;L^0_2)$. Then the term $[S_{\alpha,n}(t-\tau) - S_{\alpha}(t-\tau)]A\Psi(\tau)$ may be estimated like the difference $\Phi_{\alpha,n}$ defined by (22).

Hence, from (30) and (28), for the first term of the right hand side of (29) we obtain

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} \mathbb{E}(N^2_{n,1}(t)) \to 0.$$

For the second and third terms of (29) we can follow the same steps as above for proving (28). We have to use the properties of Yosida approximation, particularly the convergence (31). So, we can deduce that

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} \mathbb{E}|A_n W^{\Psi}_{\alpha,n}(t) - AW^{\Psi}_{\alpha}(t)|^2_H = 0,$$

what gives (19). ∎

Now, we are able to formulate the main result of this section.

Theorem 4 *Suppose that assumptions of Lemma 1 hold. Then the equation (1) with* $X_0 \equiv 0$ *has a strong solution. Precisely, the convolution* W_α^Ψ *defined by (19) is the strong solution to (1) with* $X_0 \equiv 0$.

Proof We have to show only the condition (17). By Proposition 2, the convolution $W_\alpha^\Psi(t)$, $t \geq 0$, $\alpha > 0$, has integrable trajectories, that is, $W_\alpha^\Psi(\cdot) \in L^1([0,T];H)$, P-a.s. The closed linear unbounded operator A becomes bounded on $(D(A), |\cdot|_{D(A)})$, see [12, Chapter 5]. So, we obtain $AW_\alpha^\Psi(\cdot) \in L^1([0,T];H)$, P-a.s. Hence, the function $g_\alpha(T-\tau)AW_\alpha^\Psi(\tau)$ is integrable with respect to τ, what finishes the proof. ■

The following result is an immediate consequence of Corollary 1 and Theorem 4.

Corollary 3 *Assume that* A *generates an analytic semigroup of angle* $\pi/2$ *and* $\alpha \in (0,1)$. *If* $X_0 = 0$, *then the equation (1) has a strong solution.*

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References

[1] E. Bazhlekova, *Fractional Evolution Equations in Banach Spaces,* Ph.D. Dissertation, Eindhoven University of Technology, 2001.

[2] S. Bonaccorsi, L. Tubaro, *Mittag-Leffler's function and stochastic Volterra equations of convolution type,* Stochastic Anal. Appl. **21** (1) (2003), 61-78.

[3] Ph. Clément, G. Da Prato, *Some results on stochastic convolutions arising in Volterra equations perturbed by noise*, Rend. Math. Acc. Lincei s. 9, **7** (1996), 147-153.

[4] G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*, Cambridge University Press, Cambridge, 1992.

[5] K.J. Engel, R.Nagel, *One-parameter semigroups for linear evolution equations*, Graduate texts in Mathematics **194**, Springer, New York, 2000.

[6] Y. Fujita, *Integrodifferential equations which interpolates the heat equation and the wave equation*, J. Math. Phys. **30** (1989), 134–144.

[7] H. Holden, B. Øksendal, J. Ubøe, T. Zhang, *Stochastic Partial Differential Equations: A modeling, white noise functional approach,* Probability and its applicatons, Birkhäuser, 1996.

[8] A. Karczewska, *Properties of convolutions arising in stochastic Volterra equations*, preprint: http://xxx.lanl.gov/ps/math.PR/0509012

[9] A. Karczewska, C. Lizama, *Strong solutions to stochastic Volterra equations*, submitted.

[10] M. Li, Q. Zheng, *On spectral inclusions and approximations of α-times resolvent families,* Semigroup Forum **69** (2004), 356-368.

[11] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1983.

[12] J. Weidmann, *Linear operators in Hilbert spaces*, Springer, New York, 1980.